Introduction to superselection sector theory I

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QFT and Topological Phases via Homotopy Theory and Operator Algebras

30 June – 3 July, 2025

Outline

1. Introduction

2. The toric code

3. Excitations

Introduction

What to expect?

Main issue: the classification of gapped ground states:

- We will always work in the thermodynamic limit
- Gapped ground states of local Hamiltonians...
- ... with some equivalence relation
- Focus on states with topological order (or long-range entanglement)
- Non-invertible states

Question

Can we find (physically interesting) invariants?

Why are these states interesting?

- Can host anyons: quasi-particles/superselection sectors/charges/... with braided statistics
- Algebraic properties of anyons are described by braided tensor C*-categories (typically even modular or braided fusion)
- 'Topological' nature makes these properties robust
- In other words, the category should be an invariant

Question

How can we obtain the category of anyons from a microscopic description of the state? (And is this indeed an invariant?)

What *not* to expect?

- Not a historical overview
- Only non-chiral topological order
- Will focus on basics, not most general statements
- Only discuss the operator-algebraic “DHR approach” to superselection sectors
- Will gloss over more technical details

Plan for the week

- Lecture 1: The toric code and its ground states
- Lecture 2&3: The category of superselection sectors
- Lecture 4: Classification of phases and long-range entanglement

We illustrate the theory by the example of the toric code, but methods work much more general!

Some history

Approach is rooted in Doplicher-Haag-Roberts theory:[1]

- Originates in algebraic quantum field theory, defined in terms of Haag-Kastler nets of observables $\mathscr{O} \mapsto \mathfrak{A}(\mathscr{O})$
- DHR theory attempts to capture 'charges' and leads to Bose/Fermi (para-)statistics in (3+1)D
- Culminates in Doplicher-Roberts theorem: a STC*-category is equivalent to Rep G for some compact group G
- In lower dimensions, can get braided statistics (anyons!)
- Similar techniques have been very successful in CFT (conformal nets)

[1] Haag, *Local Quantum Physics*, Springer (1992)

Different approaches

The main feature of the approach is the appearance of a braiding (describing anyon exchange).

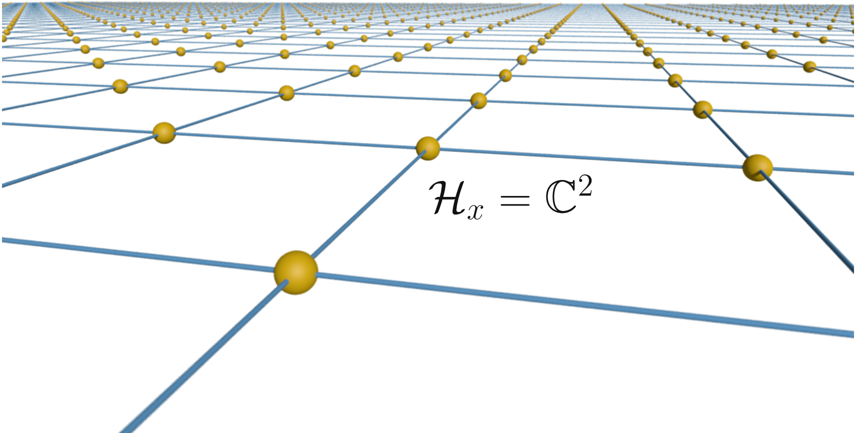
Question

How does this braiding appear?

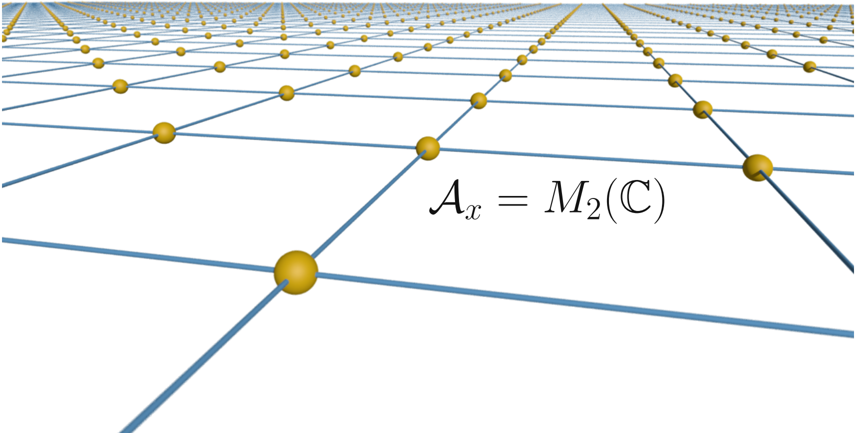
- ‘Classical DHR approach’: these lectures
  (See also Ogata, arXiv:2106.15741)
- Prefactorisation algebras (geometric approach)
  (Benini, Carmona, PN, Schenkel, arXiv:2505.07960)
  → talk Alexander Schenkel next week
- Axiomatic approach: nets on certain posets
  (Bhardwaj, Brisky, Chuah, Kawagoe, Keslin, Penneys, Wallick: arXiv:2410.21454)

The toric code

The toric code



The toric code



Pauli matrices

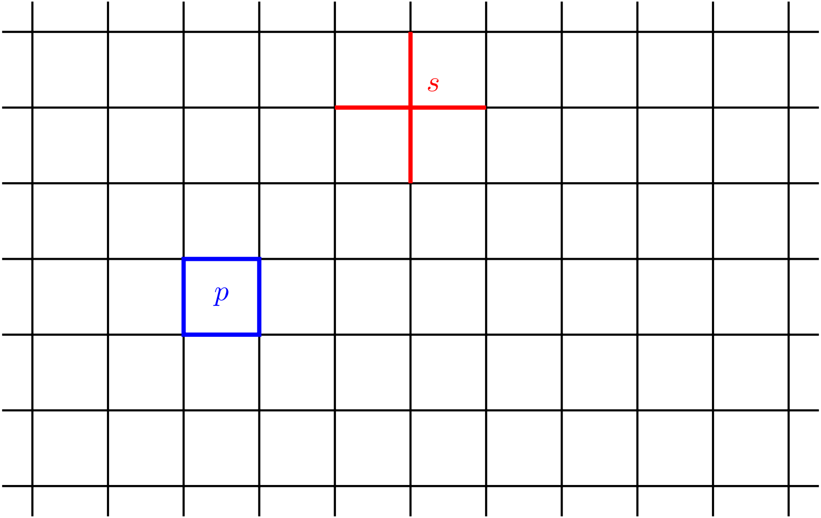
Recall the definition of the Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

They have nice algebraic properties: $\{\sigma^i, \sigma^j\} = 2\delta_{i,j} I$:

- Square to the identity
- Different Pauli matrices anti-commute
- Together with I form a basis of $M_2(\mathbb{C})$.

Stars and plaquettes



Dynamics

We define star and plaquette operators:

$$A_s = \bigotimes_{j \in s} \sigma_j^x, \qquad B_p = \bigotimes_{j \in p} \sigma_j^z.$$

Some easy properties: $A_s^2 = B_p^2 = I$, and all commute.
Can use this to define the dynamics:

$$H_\Lambda = \sum_{s \subset \Lambda} (I - A_s) + \sum_{p \subset \Lambda} (I - B_p)$$

Note that the dynamics are very simple ("commuting projector")!

Frustration-free ground state

Lemma

Let $X_i \leq I$ be a set of operators and suppose that there is a unique state ω such that $\omega(X_i) = 1$ for all X_i. Then ω is pure.

Proof.

Let ϕ be a positive linear functional such that $\phi \leq \omega$. Since $I - X_i \geq 0$, we have

$$0 \leq \phi(I - X_i) \leq \omega(I - X_i) = 0.$$

Hence $\phi(X_i) = \phi(I)$. From the uniqueness assumption, $\phi = \phi(I)\omega$, and it follows that ω is pure. □

Lemma

Let $X \leq I$ with $\omega(X) = 1$. Then $\omega(A) = \omega(AX) = \omega(XA)$.

Frustration-free ground state

Theorem

The toric code has a unique frustration free ground state ω_0. This state is pure.

Proof.

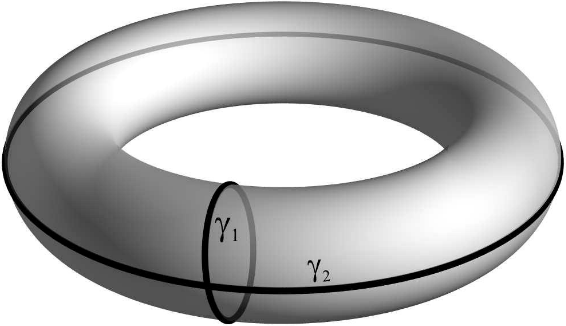
One can show (exercise!) that there is a state such that $\omega_0(A_s) = \omega_0(B_p) = 1$ for all star and plaquette operators, and these conditions uniquely determine it. Hence ω_0 is pure. Note that $\omega_0(I - A_s) = \omega_0(I - B_p) = 0$. For $A \in \mathfrak{A}_{\mathrm{loc}}$, we have

$$\begin{aligned} -i\omega_0(A^*\delta(A)) &= \sum_s \omega_0(A^*AA_s) - \omega_0(A^*A_sA) + B_p \textbf{ terms} \\ &= \sum_s \omega_0(A^*(I - A_s)A) + B_p \textbf{ terms} \\ &\geq 0 \end{aligned}$$

□

An aside...

When defined on a non-trivial topology (e.g. a torus), the condition $\omega(A_s) = \omega(B_p) = 1$ fixes the state locally but not globally. In fact, the ground state space is a quantum error correction code!



Ground space degeneracy is given by 4^g

GNS representation

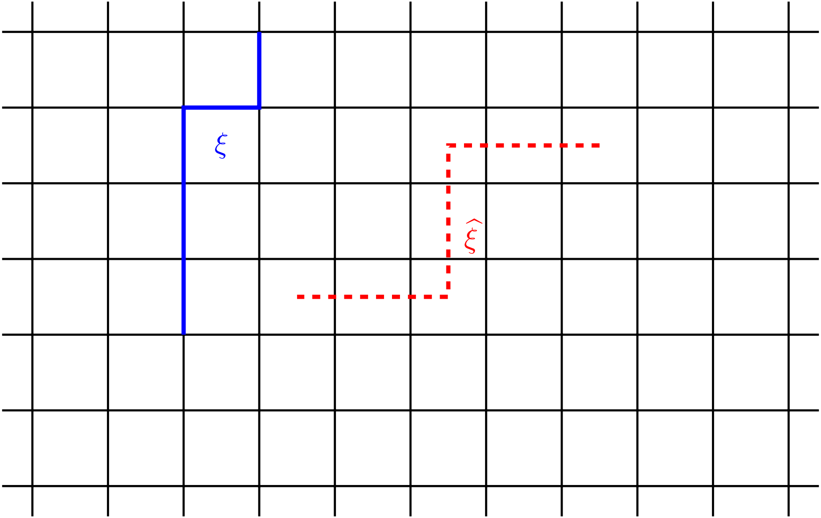
We will use ω_0 throughout as a reference state.

- GNS representation $(\pi_0, \Omega, \mathcal{H}_0)$
  - $\pi_0 : \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_0)$ $*$-representation
  - $\pi_0(\mathfrak{A})\Omega$ dense in $\mathcal{H}_0$
  - $\omega_0(A) = \langle \Omega, \pi_0(A)\Omega \rangle_{\mathcal{H}_0}$
- Will often identify $\pi_0(A)$ with A
- We have $A_s\Omega = B_p\Omega = \Omega$ (stabiliser condition):
  $\rightsquigarrow H_\Lambda \Omega = 0$ for all Λ
- Hamiltonian in GNS representation with $H\Omega = 0$, $H \geq 0$ satisfies $\mathbf{spec}(H) \cap (0,2) = \emptyset$ (spectral gap)
- State satisfies LTQO conditions: spectral gap is stable!

Excitations

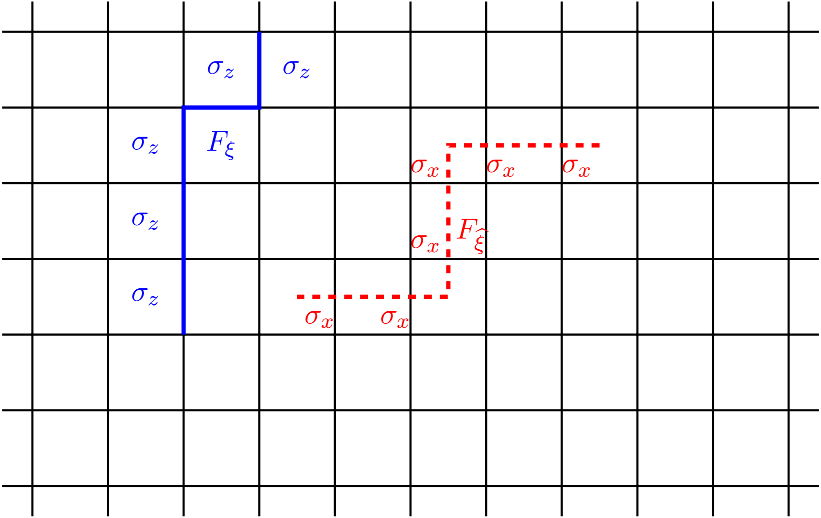
Path operators

We can consider paths ξ and dual paths $\widehat{\xi}$:



Path operators

And corresponding operators F_ξ and $F_{\widehat{\xi}}$:



Path operators

The edges on which the path operators act always have an even number in common with star and plaquette operators and hence

$$[F_\xi, A_s] = [F_\xi, B_p] = [F_{\widehat{\xi}}, A_s] = [F_{\widehat{\xi}}, B_p] = 0$$

except at the endpoints of the path! Path operators F_ξ anti-commute with star operators at endpoint, whilst the $F_{\widehat{\xi}}$ anti-commute with the plaquette operators.

Excitations

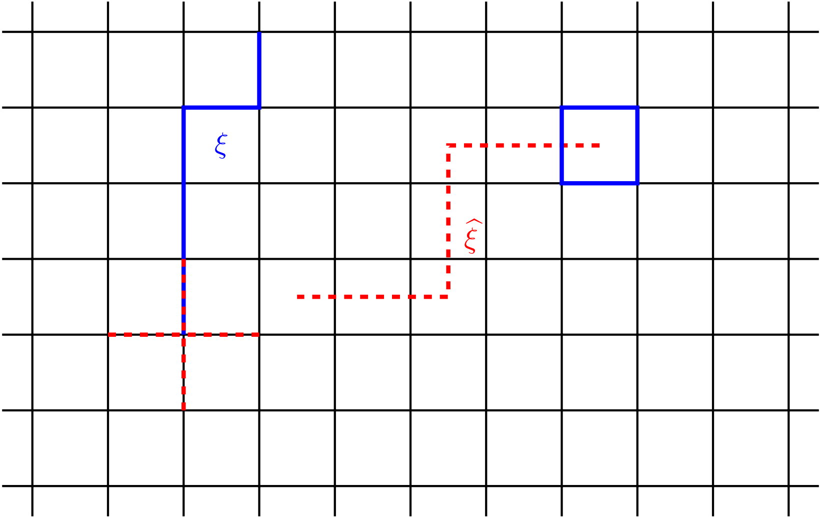
The path operators create a pair of electric or magnetic excitations respectively.

We have

$$H_\Lambda F_\xi \Omega = 2\#(\partial\xi \cap \Lambda) F_\xi \Omega, \qquad H_\Lambda F_{\widehat{\xi}} \Omega = 2\#(\partial\widehat{\xi} \cap \Lambda) F_{\widehat{\xi}} \Omega$$

where $\#(\partial\xi \cap \Lambda)$ is the number of endpoints of ξ within Λ

Path operators



Single excitations

The state $F_\xi\Omega$ describes a pair of anyons/excitations. Alternatively, in the Heisenberg picture,

$$\rho_\xi^Z(a) := F_\xi a F_\xi^* = (\mathbf{Ad}\, F_\xi)(a)$$

is an automorphism of $\mathfrak{A}$ that describes how observables change in the presence of the two excitations.

Question

Can we describe a *single* excitation?

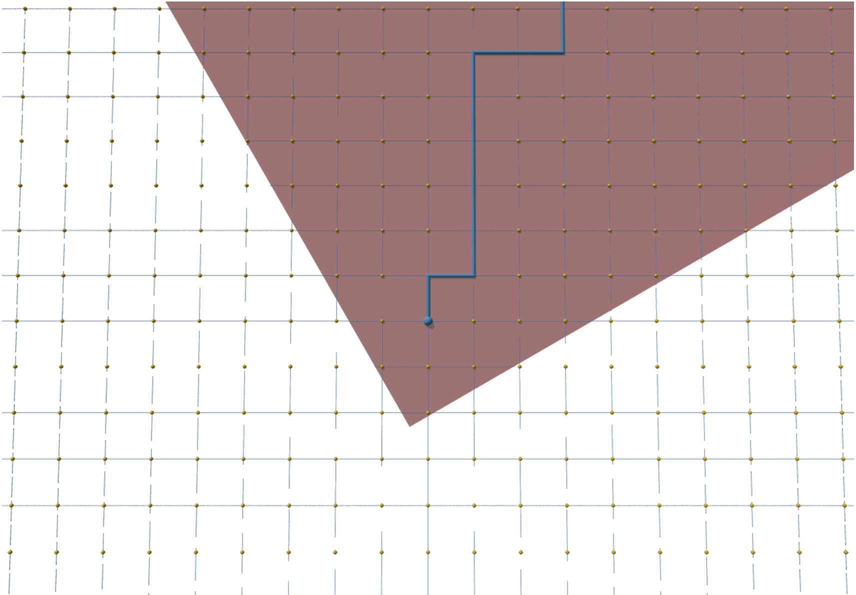
Answer

Yes! We work on an infinite lattice, so can send one of the excitations to infinity:

$$\rho_\xi^Z(a) := \lim_{n\to\infty} F_{\xi_n} a F_{\xi_n}^*$$

where ξ_n are the first n parts of a semi-infinite ribbon ξ.

Cones



Localised automorphisms

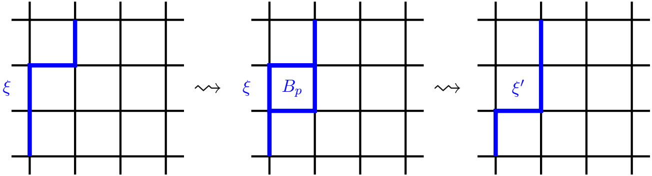
- Can choose a cone Λ as in the picture...
- ... and a semi-infinite path $\xi \subset \Lambda$.
- We get a corresponding automorphism ρ_ξ^Z.
- This is localised in Λ: $\rho_\xi^Z(a) = a$ for all $a \in \mathfrak{A}(\Lambda^c)$.

Notation

We can do something similar for dual paths to get ρ_ξ^X or for the combination of a path and dual path, to get ρ_ξ^Y. We will use the notation ξ for all types of paths, with notation ρ_ξ^k for $k = X, Y, Z$. By definition, $\rho_\xi^0 = \mathbf{id}$.

Single anyon states

The states $\omega_0 \circ \rho_\xi^k$ describe single anyon states (trivial, electric, magnetic, and "combined" state)!
These have a topological property, in the sense that the "direction" of the string is invisible:



So we have

$$\omega_0(F_\xi a F_\xi^*) = \omega_0(B_p F_\xi a F_\xi^* B_p) = \omega_0(F_{\xi'} a F_{\xi'}^*)$$

We may write $\omega_0 \circ \rho_x^k$ where x is the endpoint of the path. Note that the automorphism ρ^k do depend on ξ!

Equivalence of states on $\mathfrak{A}$

Definition

Let ω_1, ω_2 be two pure states. Then we say they are equivalent if the corresponding GNS representations are unitarily equivalent.

Lemma

Two pure states ω_1 and ω_2 on the quasi-local algebra $\mathfrak{A}$ are equivalent if and only if for every $\epsilon > 0$, there is some finite set Λ such that for every local observable A localised outside Λ we have

$$|\omega_1(A) - \omega_2(A)| \leq \epsilon \|A\|.$$

Pure states are inequivalent if they can be distinguished 'at infinity'!

Inequivalence of states

Lemma

The states $\omega_0 \circ \rho_x^k$ *and* $\omega_0 \circ \rho_y^{k'}$ *are inequivalent if* $k \neq k'$.

Proof.

We can move the excitations over a finite distance using local unitaries, so wlog we may assume $x = y$. Consider a closed loop ξ. Then one sees that

$$F_\xi = \prod_{p \subset \mathbf{int}(\xi)} B_p,$$

and something similar for closed dual loops.
Since the ribbon operators commute with any A_s and B_p (apart from possibly at the end-point, where they may anti-commute), it follows that $\rho_x^k(F_\xi) = \pm F_\xi$, and $\omega_0 \circ \rho_x^k(F_\xi) = \pm 1$.

Inequivalence of states

Lemma

The states $\omega_0 \circ \rho_x^k$ and $\omega_0 \circ \rho_y^{k'}$ are inequivalent if $k \neq k'$.

Proof.

(... cont.) The result is -1 only if ξ circles around x, and $k \neq 0$ and ξ is of a different type! Since for any finite set Λ, we can choose a loop surrounding Λ and the endpoint of the semi-infinite ribbon, we can always find an operator $X \in \mathfrak{A}(\Lambda^c)$ with $\|X\| = 1$ such that

$$|\omega_0 \circ \rho_x^k(X) - \omega_0 \circ \rho_x^{k'}(X)| = 2.$$

Since both states are pure, the result follows. □

Some references

[FN15] Leander Fiedler and Pieter Naaijkens, *Haag duality for Kitaev's quantum double model for abelian groups*, Rev. Math. Phys. 27 (2015), no. 9, 1550021, 43.

[Hal06] Hans Halvorson, *Algebraic quantum field theory*, Philosophy of Physics (Jeremy Butterfield and John Earman, eds.), Elsevier, 2006, pp. 731–922.

[Naa17] Pieter Naaijkens, *Quantum spin systems on infinite lattices: a concise introduction*, Lecture Notes in Physics, vol. 933, Springer, Cham, 2017.

[Oga22] Yoshiko Ogata, *A derivation of braided C*-tensor categories from gapped ground states satisfying the approximate Haag duality*, Journal of Mathematical Physics 63 (2022), no. 1, 011902.